PRODUCT OF STATISTICAL MANIFOLDS WITH A NON-DIAGONAL METRIC

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(Communicated by the associate editor name)

Abstract. In this paper, we generalize the dualistic structures on warped product manifolds to the dualistic structures on generalized warped product manifolds. we develop an expression of curvature for the connection of the generalized warped product in relation to those corresponding analogues of its base and fiber and warping functions. we show that the dualistic structures on the base M_1 and the fiber M_2 induces a dualistic structure on the generalized warped product $M_1 \times M_2$ and conversely, moreover, $(M_1 \times M_1, G_{f_1 f_2})$ or $(M_1 \times M_1, \tilde{g}_{f_1 f_2})$ is statistical manifold if and only if (M_1, g_1) and (M_1, g_1) are. Finally, Some interesting consequences are also given.

1. **Introduction.** The warped product provides a way to construct new pseudo-rieman nian manifolds from the given ones, see [8],[4] and [3]. This construction has useful applications in general relativity, in the study of cosmological models and black holes. It generalizes the direct product in the class of pseudo-Riemannian manifolds and it is defined as follows. Let (M_1, g_1) and (M_2, g_2) be two pseudo-Riemannian manifolds and let $f_1 : M_1 \longrightarrow \mathbb{R}^*$ be a positive smooth function on M_1, the warped product of (M_1, g_1) and (M_2, g_2) is the product manifold $M_1 \times M_2$ equipped with the metric tensor $g_{f_1} := \pi_1^* g_1 + (f \circ \pi_1)^2 \pi_2^* g_2$, where π_1 and π_2 are the projections of $M_1 \times M_2$ onto M_1 and M_2 respectively. The manifold M_1 is called the base of $(M_1 \times M_2, g_{f_1})$ and M_2 is called the fiber. The function f_1 is called the warping function.

The doubly warped product is construction in the class of pseudo-Riemannian manifolds generalizing the warped product and the direct product, it is obtained by homothetically distorting the geometry of each base $M_1 \times \{q\}$ and each fiber $\{p\} \times M_2$ to get a new "doubly warped" metric tensor on the product manifold and defined as follows. For $i \in \{1, 2\}$, let M_i be a pseudo-Riemannian manifold equipped with metric g_i, and $f_i : M_i \to \mathbb{R}^*$ be a positive smooth function on M_i. The well-know notion of doubly warped product manifold $M_1 \times_{f_1 f_2} M_2$ is defined as the product manifold $M = M_1 \times M_2$ equipped with pseudo-Riemannian metric which is denoted by $g_{f_1 f_2}$, given by

$$g_{f_1 f_2} = (f_2 \circ \pi_2)^2 \pi_1^* g_1 + (f_1 \circ \pi_1)^2 \pi_2^* g_2 .$$

2010 *Mathematics Subject Classification.* 53B05, 53C15, 53C42, 53C50.

Key words and phrases. conjugate; dual connection; product manifold; warped product; generalized warped products.

The generalized warped product is defined as follows. let c be an arbitrary real number and let g_i, $(i = 1, 2)$ be a Riemannian metric tensors on M_i. Given a smooth positive function f_i on M_i, the generalized warped product of (M_1, g_1) and (M_2, g_2) is the product manifold $M_1 \times M_2$ equipped with the metric tensor $G_{f_1 f_2}$ (see [6]), explicitly, given by

$$\begin{aligned} G_{f_1,f_2}(X,Y) &= (f_2^v)^2 g_1^{\pi_1}(d\pi_1(X), d\pi_1(Y)) + (f_1^h)^2 g_2^{\pi_2}(d\pi_2(X), d\pi_2(Y)) \\ &\quad + c f_1^h f_2^v \left(X(f_1^h) Y(f_2^v) + X(f_2^v) Y(f_1^h) \right). \end{aligned}$$

For all $X, Y \in \Gamma(TM_1 \times M_2)$. When the warping functions $f_1 = 1$ or $f_2 = 1$ or $c = 0$ we obtain a warped product or direct product.

Dualistic structures are closely related to statistical mathematics. They consist of pairs of affine connections on statistical manifolds, compatible with a pseudo-Riemanniann metric [1]. Their importance in statistical physics was underlined by many authors: [5],[2] etc.
Let M be a pseudo-Riemannian manifold equipped with a pseudo-Riemannian metric g and let ∇, ∇^* be the affine connections on M. We say that a pair of affine connections ∇ and ∇^* are compatible (or conjugate) with respect to g if

$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z) \text{ for all } X, Y, Z \in \Gamma(TM), \tag{1}$$

where $\Gamma(TM)$ is the set of all tangent vector fields on M. Then the triplet (g, ∇, ∇^*) is called the dualistic structure on M.
We note that the notion of "conjugate connection " has been attributed to A.P. Norden in affine differential geometry literarture (Simon, 2000) and was independently introduced by (Nagaoka and Amari, 1982) in information geometry, where it was called " dual connection" (Lauritzen, 1987). The triplet (M, ∇, g) is called a statistical manifold if it admits another torsion-free connection ∇^* satisfying the equation (1). We call ∇ and ∇^* duals of each other with respect to g.

In the notions of terms on statistical manifolds, for a torsion-free affine connection ∇ and a pseudo-Riemannian metric g on a manifold M, the triple (M, ∇, g) is called a statistical manifold if ∇g is symmetric. If the curvature tensor R of ∇ vanishes, (M, ∇, g) is said to be flat.

This paper extends the study of dualistic structures on warped product manifolds, [9], to dualistic structures on generalized warped products in pseudo-Riemannian manifolds. We develop an expression of curvature for the connection of the generalized warped product in relation to those corresponding analogues of its base and fiber and warping functions.
The paper is organized as follows. In section 2, we collect the basic material about Levi-Civita connection, the notion of conjugate, horizontal and vertical lifts and the generalized warped products.
In section 3, we show that the projection of a dualistic structure defined on a generalized warped product space $(M_1 \times M_2, G_{f_1 f_2})$ induces dualistic structures on the base (M_1, g_1) and the fiber (M_2, g_2). Conversely, there exists a dualistic structure on the generalized warped product space induced by its base and fiber.
In section 4, we show that the projection of a dualistic structure defined on a generalized warped product space $(M_1 \times M_2, \tilde{g}_{f_1 f_2})$ induces dualistic structures on the

base (M_1, g_1) and the fiber (M_2, g_2). Conversely, there exists a dualistic structure on the generalized warped product space induced by its base and fiber and finally, Some interesting consequences are also given.

2. **Preliminaries.**

2.1. **Statistical manifolds.**

We recall some standard facts about Levi-Civita connections and the dual statistical manifold. Many fundamental definitions and results about dualistic structure can be found in Amari's monograph ([1],[2]).

Let (M, g) be a pseudo-Riemannian manifold. The metric g defines the musical isomorphisms

$$\begin{array}{cccc} \sharp_g : & T^*M & \to & TM \\ & \alpha & \mapsto & \sharp_g(\alpha) \end{array}$$

such that $g(\sharp_g(\alpha), Y) = \alpha(Y)$, and its inverse $\flat_g$. We can thus define the cometric $\widetilde{g}$ of the metric g by :

$$\widetilde{g}(\alpha, \beta) = g(\sharp_g(\alpha), \sharp_g(\beta)). \tag{2}$$

A fundamental theorem of pseudo-Riemannian geometry states that given a pseudo-Riemannian metric g on the tangent bundle TM, there is a unique connection (among the class of torsion-free connection) that "preserves" the metric; as long as the following condition is satisfied:

$$X(g(Y, Z)) = g(\hat{\nabla}_X Y, Z) + g(Y, \hat{\nabla}_X Z) \; for \; X, Y, Z \in \Gamma(TM) \tag{3}$$

Such a connection, denoted as $\hat{\nabla}$, is known as the Levi-Civita connection. Its component forms, called Christoffel symbols, are determined by the components of pseudo-metric tensor as ("Christoffel symbols of the second Kink ")

$$\hat{\Gamma}^k_{ij} = \sum_l \frac{1}{2} g^{kl} (\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l})$$

and ("Christoffel symbols of the first Kink")

$$\hat{\Gamma}_{ij,k} = \frac{1}{2} (\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}).$$

The Levi-Civita connection is compatible with the pseudo metric, in the sense that it treats tangent vectors of the shortest curves on a manifold as being parallel.

It turns out that one can define a kind of "Compatibility" relation more generally than expressed by the (3), by introducing the notion of "Conjugate" (denoted by *) between two affine connections.

Let (M, g) be a pseudo-Riemannian manifold and let ∇, ∇^* be an affine connections on M. A connection ∇^* is said to be "conjugate" to ∇ with respect to g if

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z) \; for \; X, Y, Z \in \Gamma(TM) \tag{4}$$

Clearly,

$$(\nabla^*)^* = \nabla.$$

Otherwise, $\hat{\nabla}$, which satisfies the (3), is special in the sense that it is self-conjugate

$$(\hat{\nabla})^* = \hat{\nabla}.$$

Because pseudo-metric tensor g provides a one-to-one mapping between vectors in the tangent space and co-vectors in the cotangent space, the equation (1) can also

be seen as characterizing how co-vector fields are to be parallel-transported in order to preserve their dual pairing $< .,. >$ with vector fields.
Writing out the equation 1 explicitly,

$$\frac{\partial g_{ij}}{\partial x^k} = \Gamma_{ki,j} + \Gamma^*_{kj,i}, \tag{5}$$

where

$$\nabla^*_{\partial_i}\partial_j = \sum_l \Gamma^{*l}_{ij}\partial_l$$

so that

$$\Gamma^*_{kj,i} = g(\nabla^*_{\partial_j}\partial_k, \partial_i) = \sum_l g_{il}\Gamma^{*l}_{kj}.$$

In the following, a manifold M with a pseudo-metric g and a pair of conjugate connections ∇, ∇^* with respect to g is called a " pseudo-Riemannian manifold with dualistic structure " and denoted by (M, g, ∇, ∇^*).
Obviously, ∇ and ∇^* (or equivalently, Γ and Γ^*) satisfy the relation

$$\hat{\nabla} = \frac{1}{2}(\nabla + \nabla^*) \text{ (or equivalently, } \hat{\Gamma} = \frac{1}{2}(\Gamma + \Gamma^*)).$$

Thus an affine connection ∇ on (M, g) is metric if and only if $\nabla^* = \nabla$ (that it is self-conjugate).
For a torsion-free affine connection ∇ and a pseudo-Riemannian metric g on a manifold M, the triplet (M, ∇, g) is called a statistical manifold if ∇g is symmetric. If the curvature tensor $\mathcal{R}$ of ∇ vanishes, (M, ∇, g) is said to be flat.
For a statistical manifold (M, ∇, g), the conjugate connection ∇^* with respect to g is torsion-free and $\nabla^* g$ symmetric. Then the triplet (M, ∇^*, g) is called the dual statistical manifold of (M, ∇, g) and (∇, ∇^*, g) the dualistic structure on M. The curvature tensor of ∇ vanishes if and only if that of ∇^* does and in such a case, (∇, ∇^*, g) is called the dually flat structure [2].
More generally, in information geometry, a one-parameter family of affine connections $\nabla^{(\lambda)}$ indexed by λ ($\lambda \in \mathbb{R}$), called $\lambda-$ connections, is introduced by Amari and Nagaoka in ([1],[2]).

$$\nabla^{(\lambda)} = \frac{1+\lambda}{2}\nabla + \frac{1-\lambda}{2}\nabla^* \text{ (or equivalently, } \Gamma^{(\lambda)} = \frac{1+\lambda}{2}\Gamma + \frac{1-\lambda}{2}\Gamma^*). \tag{6}$$

Obviously, $\nabla^{(0)} = \hat{\nabla}$.
It can be shown that for a pair of conjugate connections ∇, ∇^*, their curvature tensors R, $\mathcal{R}^*$ satisfy

$$g(\mathcal{R}(X,Y)Z, W) + g(Z, \mathcal{R}^*(X,Y)W) = 0, \tag{7}$$

and more generally

$$g(\mathcal{R}^{(\lambda)}(X,Y)Z, W) + g(Z, \mathcal{R}^{*(\lambda)}(X,Y)W) = 0. \tag{8}$$

If the curvature tensor $\mathcal{R}$ of ∇ vanishes, ∇ is said to be flat.
So, ∇ is flat if and only if ∇^* is flat. In this case, (M, g, ∇, ∇^*) is said to be dually falt.
When ∇, ∇^* is dually flat, then $\nabla^{(\lambda)}$ is called λ-transitively flat [**?**]. In such case, $(M, g, \nabla^{(\lambda)}, \nabla^{*(\lambda)})$ is called an "λ-Hessian manifold", or a manifold with λ-Hessian structure.

2.2. **Horizontal and vertical lifts.** Throughout this paper M_1 and M_2 will be respectively m_1 and m_2 dimensional manifolds, $M_1 \times M_2$ the product manifold with the natural product coordinate system and $\pi_1 : M_1 \times M_2 \to M_1$ and $\pi_2 : M_1 \times M_2 \to M_2$ the usual projection maps.

We recall briefly how the calculus on the product manifold $M_1 \times M_2$ derives from that of M_1 and M_2 separately. For details see [8].

Let φ_1 in $C^\infty(M_1)$. The horizontal lift of φ_1 to $M_1 \times M_2$ is $\varphi_1^h = \varphi_1 \circ \pi_1$. One can define the horizontal lifts of tangent vectors as follows. Let $p_1 \in M_1$ and let $X_{p_1} \in T_{p_1}M_1$. For any $p_2 \in M_2$ the horizontal lift of X_{p_1} to (p_1, p_2) is the unique tangent vector $X^h_{(p_1,p_2)}$ in $T_{(p_1,p_2)}(M_1 \times M_2)$ such that $d_{(p_1,p_2)}\pi_1(X^h_{(p_1,p_2)}) = X_{p_1}$ and $d_{(p_1,p_2)}\pi_2(X^h_{(p_1,p_2)}) = 0$.
We can also define the horizontal lifts of vector fields as follows. Let $X_1 \in \Gamma(TM_1)$. The horizontal lift of X_1 to $M_1 \times M_2$ is the vector field $X_1^h \in \Gamma(T(M_1 \times M_2))$ whose value at each (p_1, p_2) is the horizontal lift of the tangent vector $(X_1)p_1$ to (p_1, p_2). For $(p_1, p_2) \in M_1 \times M_2$, we will denote the set of the horizontal lifts to (p_1, p_2) of all the tangent vectors of M_1 at p_1 by $L(p_1, p_2)(M_1)$. We will denote the set of the horizontal lifts of all vector fields on M_1 by $\mathfrak{L}(M_1)$.

The vertical lift φ_2^v of a function $\varphi_2 \in C^\infty(M_2)$ to $M_1 \times M_2$ and the vertical lift X_2^v of a vector field $X_2 \in \Gamma(TM_2)$ to $M_1 \times M_2$ are defined in the same way using the projection π_2. Note that the spaces $\mathfrak{L}(M_1)$ of the horizontal lifts and $\mathfrak{L}(M_2)$ of the vertical lifts are vector subspaces of $\Gamma(T(M_1 \times M_2))$ but neither is invariant under multiplication by arbitrary functions $\varphi \in C^\infty(M_1 \times M_2)$.

Observe that if $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{m_1}}\}$ is the local basis of the vector fields (resp. $\{dx_1, \dots, dx_{m_1}\}$ is the local basis of 1-forms) relative to a chart (U, Φ) of M_1 and $\{\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_{m_2}}\}$ is the local basis of the vector fields (resp. $\{dy_1, \dots, dy_{m_2}\}$ the local basis of the 1-forms) relative to a chart (V, Ψ) of M_2, then $\{(\frac{\partial}{\partial x_1})^h, \dots, (\frac{\partial}{\partial x_{m_1}})^h, (\frac{\partial}{\partial y_1})^v, \dots, (\frac{\partial}{\partial y_{m_2}})^v\}$ is the local basis of the vector fields (resp. $\{(dx_1)^h, \dots, (dx_{m_1})^h, (dy_1)^v, \dots, (dy_{m_2})^v\}$ is the local basis of the 1-forms) relative to the chart $(U \times V, \Phi \times \Psi)$ of $M_1 \times M_2$.

The following lemma will be useful later for our computations.

Lemma 2.1.

1. *Let $\varphi_i \in C^\infty(M_i)$, $X_i, Y_i \in \Gamma(TM_i)$ and $\alpha_i \in \Gamma(T^*M_i)$, $i = 1, 2$. Let $\varphi = \varphi_1^h + \varphi_2^v$, $X = X_1^h + X_2^v$ and $\alpha, \beta \in \Gamma(T^*(M_1 \times M_2))$. Then*
   i/ For all $(i, I) \in \{(1, h), (2, v)\}$, we have
   $$X_i^I(\varphi) = X_i(\varphi_i)^I, \quad [X, Y_i^I] = [X_i, Y_i]^I \quad \text{and} \quad \alpha_i^I(X) = \alpha_i(X_i)^I.$$
   ii/ If for all $(i, I) \in \{(1, h), (2, v)\}$ we have $\alpha(X_i^I) = \beta(X_i^I)$, then $\alpha = \beta$.
2. *Let ω_i and η_i be r-forms on M_i, $i = 1, 2$. Let $\omega = \omega_1^h + \omega_2^v$ and $\eta = \eta_1^h + \eta_2^v$. We have*
   $$d\omega = (d\omega_1)^h + (d\omega_2)^v \quad \text{and} \quad \omega \wedge \eta = (\omega_1 \wedge \eta_1)^h + (\omega_2 \wedge \eta_2)^v.$$

Proof. See [7]. □

Remark 1. Let X be a vector field on $M_1 \times M_2$, such that $d\pi_1(X) = \varphi(X_1 \circ \pi_1)$ and $d\pi_2(X) = \phi(X_2 \circ \pi_2)$, then $X = \varphi X_1^h + \phi X_2^v$.

2.3. **The generalized warped product.** let $\psi : M \to N$ be a smooth map between smooth manifolds and let g be a metric on k-vector bundle (F, P_F) over N. The metric $g^\psi : \Gamma(\psi^{-1}F) \times \Gamma(\psi^{-1}F) \to C^\infty(M)$ on the pull-back $(\psi^{-1}F, P_{\psi^{-1}F})$ over M is defined by

$$g^\psi(U,V)(p) = g_{\psi(p)}(U_p, V_p),\ \forall\, U,V \in \Gamma(\psi^{-1}F),\ p \in M.$$

Given a linear connection ∇^N on k-vector bundle (F, P_F) over N, the pull-back connection $\overset{\psi}{\nabla}$ is the unique linear connection on the pull-back $(\psi^{-1}F, P_{\psi^{-1}F})$ over M such that

$$\overset{\psi}{\nabla}_X \big(W \circ \psi\big) = \nabla^N_{d\psi(X)} W, \quad \forall W \in \Gamma(F),\ \forall X \in \Gamma(TM). \tag{9}$$

Further, let $U \in \psi^{-1}F$ and let $p \in M$, $X \in \Gamma(TM)$. Then

$$(\overset{\psi}{\nabla}_X U)(p) = (\nabla^N_{d_p\psi(X_p)} \widetilde{U})(\psi(p)), \tag{10}$$

where $\widetilde{U} \in \Gamma(F)$ with $\widetilde{U} \circ \psi = U$.
Now, let π_i, i=1,2, be the usual projection of $M_1 \times M_2$ onto M_i, given a linear connection $\overset{i}{\nabla}$ on vector bundle TM_i, the pull-back connection $\overset{\pi_i}{\nabla}$ is the unique linear connection on the pull-back $M_1 \times M_2 \to \pi_i^{-1}(TM_i)$ such that for each $Y_i \in \Gamma(TM_i)$, $X \in \Gamma(TM_1 \times M_2)$

$$\overset{\pi_i}{\nabla}_X \big(Y_i \circ \pi_i\big) = \overset{i}{\nabla}_{d\pi_i(X)} Y_i. \tag{11}$$

Further, let $(p_1,p_2) \in M_1 \times M_2$, $U \in \pi_i^{-1}(TM)$ and $X \in \Gamma(TM_1 \times M_2)$. Then

$$(\overset{\pi_i}{\nabla}_X U)(p_1,p_2) = \big(\overset{i}{\nabla}_{d_{(p_1,p_2)}\pi_i(X_{(p_1,p_2)})} \widetilde{U}\big)(p_i), \tag{12}$$

Now, let c be an arbitrary real number and let g_i, $(i = 1,2)$ be a Riemannian metric tensors on M_i. Given a smooth positive function f_i on M_i, the generalized warped product of (M_1, g_1) and (M_2, g_2) is the product manifold $M_1 \times M_2$ equipped with the metric tensor (see [6])

$$G_{f_1,f_2} = (f_2^v)^2 \pi_1^* g_1 + (f_1^h)^2 \pi_2^* g_2 + c f_1^h f_2^v df_1^h \odot df_2^v,$$

Where π_i, $(i = 1,2)$ is the projection of $M_1 \times M_2$ onto M_i and

$$df_1^h \odot df_2^v = df_1^h \otimes df_2^v + df_2^v \otimes df_1^h.$$

For all $X, Y \in \Gamma(TM_1 \times M_2)$, we have

$$\begin{aligned} G_{f_1,f_2}(X,Y) &= (f_2^v)^2 g_1^{\pi_1}(d\pi_1(X), d\pi_1(Y)) + (f_1^h)^2 g_2^{\pi_2}(d\pi_2(X), d\pi_2(Y)) \\ &\quad + c f_1^h f_2^v \left(X(f_1^h) Y(f_2^v) + X(f_2^v) Y(f_1^h) \right). \end{aligned}$$

It is the unique tensor fields such that for any $X_i, Y_i \in \Gamma(TM_i)$, $(i = 1,2)$

$$\tilde{g}_{f_1 f_2}(X_i^I, Y_k^K) = \begin{cases} (f_{3-i}^J)^2 g_i(X_i, Y_i)^I, & \text{if } (i,I) = (k,K) \\ c f_i^I f_k^K X_i(f_i)^I Y_k(f_k)^K, & \text{otherwise} \end{cases} \tag{13}$$

If either $f_1 \equiv 1$ or $f_2 \equiv 1$ but not both, then we obtain a singly warped product. If both $f_1 \equiv 1$ and $f_2 \equiv 1$, then we have a product manifold. If neither f_1 nor f_2 is constant and $c = 0$, then we have a nontrivial doubly warped product. If neither f_1 nor f_2 is constant and $c \neq 0$, then we have a nontrivial generalized warped product.

Now, Let us assume that (M_i, g_i), $(i = 1,2)$ is a smooth connected Riemannian manifold. The following proposition provides a necessary and sufficient condition

for a symmetric tensor field G_{f_1,f_2} of type $(0,2)$ of two Riemannian metrics to be a Riemannian metric.

Proposition 1. [6] *Let (M_i, g_i), $(i = 1, 2)$ be a Riemannian manifold and let f_i be a positive smooth function on M_i and c be an arbitrary real number. Then the symmetric tensor field $G_{f_1 f_2}$ is Riemannian metric on $M_1 \times M_2$ if and only if*

$$0 \leq c^2 g_1(gradf_1, gradf_1)^h g_2(gradf_2, gradf_2)^v < 1. \tag{14}$$

Corollary 1. [6] *If the symmetric tensor field G_{f_1,f_2} of type $(0,2)$ on $M_1 \times M_2$ is degenerate, then for any $i \in \{1,2\}$, $g_i(gradf_i, gradf_i)$ is positive constant k_i with*

$$k_i = \frac{1}{c^2 k_{(3-i)}}.$$

In all what follows, we suppose that f_1 and f_2 satisfies the inequality (14).

Lemma 2.2. [6] *Let X be an arbitrary vector field of $M_1 \times M_2$, if there exist $\varphi_i, \psi_i \in C^\infty(M_i)$ and $X_i, Y_i \in \Gamma(TM_i)$, $(i = 1, 2)$ such that*

$$\left\{ \begin{array}{l} G_{f_1 f_2}(X, Z_1^h) = G_{f_1 f_2}(\varphi_2^v X_1^h + \varphi_1^h X_2^v, Z_1^h), \\ \\ G_{f_1 f_2}(X, Z_2^v) = h^h G_{f_1 f_2}(\psi_2^v Y_1^h + \psi_1^h Y_2^v, Z_2^v). \end{array} \right. \quad \forall\ Z_i \in \Gamma(TM_i),$$

Then we have,

$$\begin{array}{rcl} X &=& \varphi_2^v X_1^h + \psi_1^h Y_2^v + c f_1^h f_2^v \left\{ \psi_2^v Y_1(f_1)^h - \varphi_2^v X_1(f_1)^h \right\} grad(f_2^v) \\ &-& c f_1^h f_2^v \left\{ \psi_1^h Y_2(f_2)^v - \varphi_1^h X_2(f_2)^v \right\} grad(f_1^h) \end{array} \tag{15}$$

3. Dualistic structure with respect to $G_{f_1 f_2}$.

Proposition 2. *Let $(\tilde{g}_{f_1 f_2}, \nabla, \nabla^*)$ be a dualistic structure on $M_1 \times M_2$. Then there exists an affine connections $\overset{i}{\nabla}$, $\overset{i}{\nabla}{}^*$ on M_i, such that $(g_i, \overset{i}{\nabla}, \overset{i}{\nabla}{}^*)$ is a dualistic structure on M_i $(i = 1, 2)$.*

Proof. Taking the affine connections on M_i, $(i = 1, 2)$.

$$\left\{ \begin{array}{ll} (\overset{1}{\nabla}_{X_1} Y_1) \circ \pi_1 = d\pi_1(\nabla_{X_1^h} Y_1^h) + c\frac{f_1^h}{f_2^v}(\nabla_{X_1^h} Y_1^h)(f_2^v)(gradf_1) \circ \pi_1, & \forall\ X_1, Y_1 \in \Gamma(TM_1) \\ (\overset{1}{\nabla}{}^*_{X_1} Y_1) \circ \pi_1 = d\pi_1(\nabla^*_{X_1^h} Y_1^h) + c\frac{f_1^h}{f_2^v}(\nabla^*_{X_1^h} Y_1^h)(f_2^v)(gradf_1) \circ \pi_1, & \\ & \\ (\overset{2}{\nabla}_{X_2} Y_2) \circ \pi_2 = d\pi_2(\nabla_{X_2^v} Y_2^v) + c\frac{f_2^v}{f_1^h}(\nabla_{X_2^v} Y_2^v)(f_1^h)(gradf_2) \circ \pi_2, & \forall\ X_2, Y_2 \in \Gamma(TM_2) \\ (\overset{2}{\nabla}{}^*_{X_2} Y_2) \circ \pi_2 = d\pi_2(\nabla^*_{X_2^v} Y_2^v) + c\frac{f_2^v}{f_1^h}(\nabla^*_{X_2^v} Y_2^v)(f_1^h)(gradf_2) \circ \pi_2. & \end{array} \right.$$

Therfore, we have for all $X_i, Y_i, Z_i \in \Gamma(TM_i)$ $(i = 1, 2)$.

$$X_i^I(G_{f_1 f_2}(Y_i^I, Z_i^I)) = G_{f_1 f_2}(\nabla_{X_i^I} Y_i^I, Z_i^I) + G_{f_1 f_2}(Y_i^I, \nabla^*_{X_i^I} Z_i^I). \tag{16}$$

Since, $d\pi_{3-i}(Z_i^I) = 0$, $X_i^I(f_{3-i}^J) = 0$ and for any $X \in \Gamma(TM_1 \times M_2)$,

$$g_{f_1 f_2}(X, Z_i^I) = (f_{3-i}^J)^2 g_i^{\pi_i}(d\pi_i(X), Z_i \circ \pi_i) + c f_1^h f_2^v X(f_{3-i}^J) Z_i(f_i)^I,$$

then the equation (23) is aquivalent to

$$(f_{3-i}^J)^2 (X_i(g_i(Y_i, Z_i)))^I = (f_{3-i}^J)^2 \big\{ g_i(\overset{i}{\nabla}_{X_i} Y_i, Z_i) + g_i(Y_i, \overset{i}{\nabla}{}^*_{X_i} Z_i) \big\}^I.$$

Where $(i, I), (3-i, J) \in \{(1, h), (2, v)\}$. Hence, the pair of affine connections $\overset{i}{\nabla}$ and $\overset{i}{\nabla}{}^*$ are conjugate with respect to g_i. $\square$

Proposition 3. *Let* $(g_i, \overset{i}{\nabla}, \overset{i}{\nabla}{}^*)$ *be a dualistic structure on* M_i $(i = 1, 2)$. *Then there exists a dualistic structure on* $M_{_1} \times M_{_2}$ *with respect to* $G_{f_1 f_2}$.

Proof. Let ∇ and ∇^* be the connections on $M_1 \times M_2$ given by

$$\left\{\begin{array}{l} d\pi_1(\nabla_X Y) = \overset{\pi_1}{\nabla}_X d\pi_1(Y) + Y(\ln f_{_2}^v) d\pi_1(X) + X(\ln f_{_2}^v) d\pi_1(Y) \\ \qquad\qquad + \frac{1}{f_1^h f_2^v (1 - c^2 b_1^h b_2^v)} \Big\{ \frac{(f_1^h)^2}{f_2^v} B_{f_{_2}^v}(X,Y) - c b_2^v f_2^v B_{f_{_1}^h}(X,Y) \\ \qquad - c f_1^h (1 - c b_2^v) \big[X(f_1^h) Y(f_2^v) + X(f_2^v) Y(f_1^h) \big] \Big\} (grad f_1) \circ \pi_1, \\ \\ d\pi_2(\nabla_X Y) = \overset{\pi_2}{\nabla}_X d\pi_2(Y) + Y(\ln f_{_1}^h) d\pi_2(X) + X(\ln f_{_1}^h) d\pi_2(Y) \\ \qquad\qquad + \frac{1}{f_1^h f_2^v (1 - c^2 b_1^h b_2^v)} \Big\{ \frac{(f_2^v)^2}{f_1^h} B_{f_{_1}^h}(X,Y) - c b_1^h f_1^h B_{f_{_2}^v}(X,Y) \\ \qquad - c f_2^v (1 - c b_1^h) \big[X(f_1^h) Y(f_2^v) + X(f_2^v) Y(f_1^h) \big] \Big\} (grad f_2) \circ \pi_2, \\ \\ d\pi_1(\nabla^*_X Y) = \overset{\pi_1}{\nabla}{}^*_X d\pi_1(Y) + Y(\ln f_{_2}^v) d\pi_1(X) + X(\ln f_{_2}^v) d\pi_1(Y) \\ \qquad\qquad + \frac{1}{f_1^h f_2^v (1 - c^2 b_1^h b_2^v)} \Big\{ \frac{(f_1^h)^2}{f_2^v} B^*_{f_{_2}^v}(X,Y) - c b_2^v f_2^v B^*_{f_{_1}^h}(X,Y) \\ \qquad - c f_1^h (1 - c b_2^v) \big[X(f_1^h) Y(f_2^v) + X(f_2^v) Y(f_1^h) \big] \Big\} (grad f_1) \circ \pi_1, \\ \\ d\pi_2(\nabla^*_X Y) = \overset{\pi_2}{\nabla}{}^*_X d\pi_2(Y) + Y(\ln f_{_1}^h) d\pi_2(X) + X(\ln f_{_1}^h) d\pi_2(Y) \\ \qquad\qquad + \frac{1}{f_1^h f_2^v (1 - c^2 b_1^h b_2^v)} \Big\{ \frac{(f_2^v)^2}{f_1^h} B^*_{f_{_1}^h}(X,Y) - c b_1^h f_1^h B^*_{f_{_2}^v}(X,Y) \\ \qquad - c f_2^v (1 - c b_1^h) \big[X(f_1^h) Y(f_2^v) + X(f_2^v) Y(f_1^h) \big] \Big\} (grad f_2) \circ \pi_2, \end{array}\right. \tag{17}$$

for any $X, Y \in \Gamma(TM_1 \times M_2)$. Where $B_{f_{_i}^I}$ and $B^*_{f_{_i}^I}$ $(i = 1, 2)$ the $(0, 2)$ tensors fields of $f_{_i}^I$ given respectively by

$$\begin{array}{rcl} B_{f_{_i}^I}(X,Y) & = & c f_{_i}^I \Big\{ X(Y(f_i^I)) - g_{_i}^{\pi_i}\big(\overset{\pi_i}{\nabla}_X d\pi_i(Y), (grad f_i) \circ \pi_i\big) \Big\} \\ & + & c X(f_{_i}^I) Y(f_{_i}^I) - \frac{1}{f_j^J} g_{_i}^{\pi_i}\big(d\pi_i(X), d\pi_i(Y)\big), \end{array}$$

and

$$\begin{array}{rcl} B^*_{f_{_i}^I}(X,Y) & = & c f_{_i}^I \Big\{ X(Y(f_i^I)) - g_{_i}^{\pi_i}\big(\overset{\pi_i}{\nabla}{}^*_X d\pi_i(Y), (grad f_i) \circ \pi_i\big) \Big\} \\ & + & c X(f_{_i}^I) Y(f_{_i}^I) - \frac{1}{f_j^J} g_{_i}^{\pi_i}\big(d\pi_i(X), d\pi_i(Y)\big), \end{array}$$

$j = i - 3$ and $(i, I), (j, J) \in \{(1, h), (2, v)\}$.
Or, for any $X_i, Y_i \in \Gamma(TM_i)$ $(i = 1, 2)$

$$\left\{\begin{array}{l} \nabla_{X_1^h} Y_1^h = (\overset{1}{\nabla}_{X_1} Y_1)^h + f_{_2}^v B_{f_1}(X_1, Y_1)^h grad(f_{_2}^v); \\ \nabla_{X_2^v} Y_2^{vh} = (\overset{2}{\nabla}_{X_2} Y_2)^v + f_{_1}^h B_{f_2}(X_2, Y_2)^v grad(f_{_1}^h); \\ \\ \nabla^*_{X_1^h} Y_1^h = (\overset{1}{\nabla}{}^*_{X_1} Y_1)^h + f_{_2}^v B^*_{f_1}(X_1, Y_1)^h grad(f_{_2}^v); \\ \nabla^*_{X_2^v} Y_2^v = (\overset{2}{\nabla}{}^*_{X_2} Y_2)^v + f_{_1}^h B^*_{f_2}(X_2, Y_2)^v grad(f_{_1}^h); \\ \\ \nabla_{X_1^h} Y_2^v = \nabla^*_{X_1^h} Y_2^v = -c X_1(f_1)^h Y_2(f_2)^v \big\{ f_{_2}^v grad(f_{_1}^h) + f_{_1}^h grad(f_{_2}^v) \big\} \\ \qquad\qquad\qquad + \big(Y_2(\ln f_{_2})\big)^v X_1^h + \big(X_1(\ln f_2)\big)^h Y_2^v \\ \nabla_{Y_2^v} X_1^h = \nabla^*_{Y_2^v} X_1^h = \nabla_{X_1^h} Y_2^v. \end{array}\right. \tag{18}$$

Where B_{f_i} and $B^*_{f_i}$ $(i=1,2)$ the $(0,2)$ tensors fields of f_i given respectively by

$$B_{f_i}(X_i,Y_i) = cf_i\left\{X_i(Y_i(f_i)) - \overset{i}{\nabla}_{X_i}Y_i(f_i)\right\} + cX_i(f_i)Y_i(f_i) - g_i(X_i,Y_i),$$

and

$$B^*_{f_i}(X_i,Y_i) = cf_i\left\{X_i(Y_i(f_i)) - \overset{i}{\nabla}{}^*_{X_i}Y_i(f_i)\right\} + cX_i(f_i)Y_i(f_i) - g_i(X_i,Y_i),$$

Let us assume that $(g_i,\overset{i}{\nabla},\overset{i}{\nabla}{}^*)$ is a dualistic structures on M_i, $i=1,2$. Let A be a tensor field of type $(0,3)$ defined for any $X,Y,Z\in\Gamma(TM_1\times M_2)$ by

$$A(X,Y,Z) = X(G_{f_1f_2}(Y,Z)) - G_{f_1f_2}(\nabla_XY,Z) - G_{f_1f_2}(Y,\nabla^*_XZ),$$

if $X_i,Y_i,Z_i\in\Gamma(TM_i)$, $i=1,2$, then we have

$$X_i^I(G_{f_1f_2}(Y_i^I,Z_i^I)) = X_i^I((f^J_{3-i})^2g_i(X_i,Y_i)^I).$$

Since $d\pi_{3-i}(X_i^I)=0$, it follows that $d\pi_{3-i}(X_i^I)(f_{3-i} = X_i^I(f^J_{3-i})=0$, and hence

$$X_i^I(G_{f_1f_2}(Y_i^I,Z_i^I)) = (f^J_{3-i})^2(X(g_i(Y_i,Z_i)))^I,$$

as $(g_i,\overset{i}{\nabla},\overset{i}{\nabla}{}^*)$ is dualistic structure, we have thus

$$X_i^I(G_{f_1f_2}(Y_i^I,Z_i^I)) = (f^J_{3-i})^2\{g_i(\overset{i}{\nabla}_{X_i}Y_i,Z_i)^I + g_i(Y_i,\overset{i}{\nabla}{}^*_{X_i}Z_i)^I\},$$

from Equations (13), (18), then it's easily seen that the following equation holds

$$A(X_i^I,Y_i^I,Z_i^I)=0$$

In the different lifts $(i\neq j)$, we have

$$X_i^I(G_{f_1f_2}(Y_i^I,Z_j^J)) = cf_j^J(Z_j(f_j))^JX_i((f_i(Y(f_i))))^I,$$

$$G_{f_1f_2}(\nabla_{X_i^J}Y_i^I,Z_j^J) = f_j^J\left\{cf_iX_i(Y_i(f_i)) + cX_i(f_i)Y_i(f_i) - g_i(X_i,Y_i)\right\}^IZ_j(f_j)^J,$$

and

$$G_{f_1f_2}(\nabla^*_{X_i^I}Z_j^J,Y_i^I) = f_j^Jg_i(X_i,Y_i)^IZ_j(f_j)^J.$$

We add these equations and obtain

$$A(X_i^I,Y_i^I,Z_j^J)=0$$

Hence the same applies for $A(X_j^J,Y_i^I,Z_i^I) = A(X_i^I,Y_j^J,Z_i^I)=0$.
This proves that ∇^* is conjugate to ∇ with respect to $G_{f_1f_2}$.

□

We recall that the connection ∇ on $M_1\times M_2$ induced by $\overset{1}{\nabla}$ and $\overset{2}{\nabla}$ on M_1 and M_2 respectively, is given by Equation (18).

Proposition 4. *$(M_1,\overset{1}{\nabla},g_1)$ and $(M_2,\overset{2}{\nabla},g_2)$ are statistical manifolds if and only if $(M_1\times M_2,G_{f_1f_2},\nabla)$ is a statistical manifold.*

Proof. Let us assume that $(M_i,\overset{i}{\nabla},g_i)$ $(i=1,2)$ is statistical manifold.
Firstly, we show that ∇ is torsion-free. Indeed; by Equation (17), we have for any $X,Y\in\Gamma(TM_1\times M_2)$

$$d\pi_i(T(X,Y)) = \overset{\pi_i}{\nabla}_Xd\pi_i(Y) - \overset{\pi_i}{\nabla}_Yd\pi_i(X) - d\pi_i([X,Y])$$

Since for $i=1,2$, $\overset{i}{\nabla}$ is torsion-free, then

$$\overset{\pi_i}{\nabla}_X d\pi_i(Y) - \overset{\pi_i}{\nabla}_Y d\pi_i(X) = d\pi_i([X,Y])$$

Therefore, from Remark 1, the connection ∇ is torsion-free.
Secondly, we show that $\nabla G_{f_1,f_2}$ is symmetric. In fact; for $i=1,2$,

$$(\nabla G_{f_1 f_2})(X_i^I, Y_i^I, Z_i^J) = X_i^I(G_{f_1 f_2}(Y_i^I, Z_i^I)) - G_{f_1 f_2}(\nabla_{X_i^I} Y_i^I, Z_i^I) - G_{f_1 f_2}(Y_i^I, \nabla_{X_i^I} Z_i^I)$$

by Equations (13), (18) and since $(\overset{i}{\nabla} g_i)$, $i=1,2$, is symmetric, we have

$$\begin{aligned}(\nabla G_{f_1 f_2})(X_i^I, Y_I^I, Z_i^I) &= (f_{3-i}^J)^2((\overset{i}{\nabla} g_i)(X_i, Y_i, Z_i))^I \\ &= (f_{3-i}^J)^2((\overset{i}{\nabla} g_i)(Y_i, X_i, Z_i))^h \\ &= (\nabla G_{f_1 f_2})(Y_i^I, X_I^I, Z_i^I).\end{aligned}$$

In the different lifts, we have

$$(\nabla G_{f_1 f_2})(X_i^I, Y_i^I, Z_{3-i}^J) = (\nabla G_{f_1 f_2})(X_{3-i}^J, Y_i^I, Z_i^I) = (\nabla G_{f_1 f_2})(X_i^I, Y_{3-i}^I, Z_i^I) = 0,$$

Therefore, $(\nabla G_{f_1 f_2})$ is symmetric. Thus $(M_1 \times M_2, g_{f_1 f_2}, \nabla)$ is a statistical manifold.

Conversely, if $(M_1 \times M_2, G_{f_1 f_2}, \nabla)$ is statistical manifold, then $(\nabla G_{f_1 f_2})$ is symmetric and ∇ is torsion-free, particularly, when $X_i, Y_i, Z_i \in \Gamma(TM_i)$, we have

$$\left\{\begin{aligned}(\nabla G_{f_1 f_2})(X_i^I, Y_I^I, Z_i^I) &= (\nabla G_{f_1 f_2})(Y_i^I, X_I^I, Z_i^I), \\ T(X_i^I, Y_i^I) &= 0.\end{aligned}\right. \quad \forall\ i=1,2,$$

Then, by Equations (13) and (18), we obtained, for $i=1,2$, $\overset{i}{\nabla} g_i$, is symmetric and $\overset{i}{\nabla}$, is torsion-free. Therefore, $(M_i, \overset{i}{\nabla}, g_i)$, $i=1,2$, is statistical manifold. □

4. **Dualistic structure with respect to $\tilde{g}_{f_1 f_2}$.** Let c be an arbitrary real number and let g_i, $(i=1,2)$ be a Riemannian metric tensors on M_i. Given a smooth positive function f_i on M_i, we define a metric tensor field on $M_1 \times M_2$ by

$$\tilde{g}_{f_1, f_2} = \pi_1^* g_1 + (f_1^h)^2 \pi_2^* g_2 + \frac{c^2}{2}(f_2^v)^2 df_1^h \odot df_1^h. \tag{19}$$

Where π_i, $(i=1,2)$ is the projection of $M_1 \times M_2$ onto M_i (see [6]).
For all $X, Y \in \Gamma(TM_1 \times M_2)$, we have

$$\tilde{g}_{f_1, f_2}(X,Y) \quad = g_1^{\pi_1}(d\pi_1(X), d\pi_1(Y)) + (f_1^h)^2 g_2^{\pi_2}(d\pi_2(X), d\pi_2(Y)) + (cf_2^v)^2 X(f_1^h) Y(f_1^h)).$$

It is the unique tensor fields such that for any $X_i, Y_i \in \Gamma(TM_i)$, $(i=1,2)$

$$\left\{\begin{array}{l}\tilde{g}_{f_1 f_2}(X_1^h, Y_1^h) = g_1(X_1, Y_1)^h + (cf_2^v)^2 X_1(f_1) Y_1(f_1)^h, \\ \tilde{g}_{f_1 f_2}(X_1^h, Y_2^v) = \tilde{g}_{f_1 f_2}(Y_2^v, X_1^h) = 0, \\ \tilde{g}_{f_1 f_2}(X_2^v, Y_2^v) = (f_1^h)^2 g_2(X_2, Y_2)^v.\end{array}\right. \tag{20}$$

Proposition 5. *Let $(\tilde{g}_{f_1 f_2}, \nabla, \nabla^*)$ be a dualistic structure on $M_1 \times M_2$. Then there exists an affine connections $\overset{i}{\nabla}$, $\overset{i}{\nabla}{}^*$ on M_i, such that $(g_i, \overset{i}{\nabla}, \overset{i}{\nabla}{}^*)$ is a dualistic structure on M_i $(i=1,2)$.*

Proof. Taking the affine connections on M_i, $(i = 1, 2)$.

$$\begin{cases} (\overset{1}{\nabla}_{X_1} Y_1) \circ \pi_1 = d\pi_1(\nabla_{X_1^h} Y_1^h) + (cf_2^v)^2 H^{f_1^h}(X_1^h, Y_1^h)(gradf_1) \circ \pi_1, \\ (\overset{1}{\nabla}{}^*_{X_1} Y_1) \circ \pi_1 = d\pi_1(\nabla^*_{X_1^h} Y_1^h) + (cf_2^v)^2 H^{*f_1^h}(X_1^h, Y_1^h)(gradf_1) \circ \pi_1, \end{cases} \tag{21}$$

$$\begin{cases} (\overset{2}{\nabla}_{X_2} Y_2) \circ \pi_2 = \frac{1}{(f_1^h)^2} d\pi_2(\nabla_{X_2^v} Y_2^v) \\ (\overset{2}{\nabla}{}^*_{X_2} Y_2) \circ \pi_2 = \frac{1}{(f_1^h)^2} d\pi_2(\nabla^*_{X_2^v} Y_2^v). \end{cases} \tag{22}$$

Therfore, we have for all $X_i, Y_i, Z_i \in \Gamma(TM_i)$ $(i = 1, 2)$.

$$X_i^I(\tilde{g}_{f_1 f_2}(Y_i^I, Z_i^I)) = \tilde{g}_{f_1 f_2}(\nabla_{X_i^I} Y_i^I, Z_i^I) + \tilde{g}_{f_1 f_2}(Y_i^I, \nabla^*_{X_i^I} Z_i^I). \tag{23}$$

Since, $d\pi_{3-i}(Z_i^I) = 0$, $X_i^I(f_{3-i}^J) = 0$ and for any $X \in \Gamma(TM_1 \times M_2)$,

$$\tilde{g}_{f_1 f_2}(X, Z_i^I) = \begin{cases} g_1^{\pi_1}(d\pi_1(X), Z_1 \circ \pi_1) + (cf_2^v)^2 X(f_1^h) Z_i(f_1)^h, & if (i, I) = (1, h) \\ (f_1^h)^2 g_2^{\pi_2}(d\pi_2(X), Z_2 \circ \pi_2), & (i, I) = (2, v) \end{cases}$$

Substituting from Equations (21) and (22) into Formula (23) gives

$$\begin{cases} (X_1(g_1(Y_1, Z_1)))^h = g_1^{\pi_1}(\overset{1}{\nabla}_{X_1} Y_1, Z_1 \circ \pi_1) + g_1^{\pi_1}(\overset{1}{\nabla}{}^*_{X_1} Z_1, Y_1 \circ \pi_1), \\ (f_1^h)^2 \left(X_2(g_2(Y_2, Z_2))\right)^v = (f_1^h)^2 \left\{ g_2^{\pi_2}(\overset{2}{\nabla}_{X_2} Y_2, Z_2 \circ \pi_2) + g_2^{\pi_2}(\overset{2}{\nabla}{}^*_{X_2} Z_2, Y_2 \circ \pi_2) \right\}, \end{cases}$$

Hence, the pair of affine connections $\overset{i}{\nabla}$ and $\overset{i}{\nabla}{}^*$ are conjugate with respect to g_i. ☐

Proposition 6. *Let $(g_i, \overset{i}{\nabla}, \overset{i}{\nabla}{}^*)$ be a dualistic structure on M_i $(i = 1, 2)$. Then there exists a dualistic structure on $M_1 \times M_2$ with respect to $\tilde{g}_{f_1 f_2}$.*

Proof. Let ∇ and ∇^* be the connections on $M_1 \times M_2$ given by

$$\begin{cases} \nabla_{X_1^h} Y_1^h &= (\overset{1}{\nabla}_{X_1} Y_1)^h + \frac{(cf_2^v)^2 H^{f_1}(X_1, Y_1)^h}{1 + (cf_2^v)^2 b_1^h} (gradf_1)^h \\ & - \; c^2 f_2^v (X_1(\ln f_1) Y_1(\ln f_1))^h (gradf_2)^v, \\ \nabla_{X_2^v} Y_2^{vh} &= (\overset{2}{\nabla}_{X_2} Y_2)^v - \frac{f_1^h g_2(X_2, Y_2)^v}{1 + (cf_2^v)^2 b_1^h} (gradf_1)^h, \\ \nabla^*_{X_1^h} Y_1^h &= (\overset{1}{\nabla}{}^*_{X_1} Y_1)^h + \frac{(cf_2^v)^2 H^{*f_1}(X_1, Y_1)^h}{1 + (cf_2^v)^2 b_1^h} (gradf_1)^h \\ & - \; c^2 f_2^v (X_1(\ln f_1) Y_1(\ln f_1))^h (gradf_2)^v, \\ \nabla^*_{X_2^v} Y_2^v &= (\overset{2}{\nabla}{}^*_{X_2} Y_2)^v - \frac{f_1^h g_2(X_2, Y_2)^v}{1 + (cf_2^v)^2 b_1^h} (gradf_1)^h, \\ \nabla_{X_1^h} Y_2^v &= \nabla^*_{X_1^h} Y_2^v = \frac{c^2 f_2^v Y_2(f_2)^v X_1(f_1)^h}{1 + (cf_2^v)^2 b_1^h} (gradf_1)^h + \left(X_1(\ln f_1)\right)^h Y_2^v, \\ \nabla_{Y_2^v} X_1^h &= \nabla^*_{Y_2^v} X_1^h = \nabla_{X_1^h} Y_2^v. \end{cases} \tag{24}$$

for any $X_i, Y_i \in \Gamma(TM_i)$ $(i = 1, 2)$. Where H^{f_1} and H^{*f_1} are the Hessian of f_1 with respect to $\overset{1}{\nabla}$ and $\overset{1}{\nabla}{}^*$ respectively.

Let us assume that $(g_i, \overset{i}{\nabla}, \overset{i}{\nabla}{}^*)$ is a dualistic structures on M_i, $i = 1, 2$. Let A be a tensor field of type $(0, 3)$ defined for any $X, Y, Z \in \Gamma(TM_1 \times M_2)$ by

$$A(X, Y, Z) = X(\tilde{g}_{f_1 f_2}(Y, Z)) - \tilde{g}_{f_1 f_2}(\nabla_X Y, Z) - \tilde{g}_{f_1 f_2}(Y, \nabla^*_X Z),$$

Since $d\pi_{3-i}(X_i^I)=0$, it follows that

$$X_i^I(f_{3-i}^J)=d\pi_{3-i}(X_i^I)(f_{3-i})=0, \qquad \forall (i,I),(j,J)\in\{(i,h),(2,v)\},$$

and hence, for all $X_i,Y_i,Z_i\in\Gamma(TM_i)$ $(i=1,2)$, we have

$$\begin{cases} X_1^h\left(\tilde g_{f_1f_2}(Y_1^h,Z_1^h)\right)=\left(X_1(g_1(Y_1,Z_1))\right)^h+(cf_2^v)^2\left\{Y_1(f_1)X_1(Z_1(f_1))+Z_1(f_1)X_1(Y_1(f_1))\right\}^h, \\ X_2^v\left(\tilde g_{f_1f_2}(Y_2^v,Z_2^v)\right)=(cf_2^v)^2\left(X_2(g_2(Y_2,Z_2))\right)^h. \end{cases}$$

as $(g_i,\overset{i}{\nabla},\overset{i}{\nabla}{}^*)$ is dualistic structure and from Equations (20), (24), then it's easily seen that the following equation holds

$$A(X_i^I,Y_i^I,Z_i^I)=0, \qquad \forall (i,I),(j,J)\in\{(i,h),(2,v)\}.$$

In the different lifts $(i\neq j)$, we have

$$X_i^I(\tilde g_{f_1f_2}(Y_i^I,Z_j^J))=0,$$

$$\begin{cases} \tilde g_{f_1f_2}(\nabla_{X_1^h}Y_1^h,Z_2^v)=-c^2f_2^vX_1(f_1)^hY_1(f_1)^hZ_2(f_2)^v, \\ \tilde g_{f_1f_2}(\nabla_{X_2^v}Y_2^v,Z_1^h)=-f_1^hg_2(X_2,Y_2)^vZ_1(f_1)^h, \end{cases}$$

and

$$\begin{cases} \tilde g_{f_1f_2}(Y_1^h,\overset{*}{\nabla}_{X_1^h}Z_2^v)=c^2f_2^vX_1(f_1)^hY_1(f_1)^hZ_2(f_2)^v, \\ \tilde g_{f_1f_2}(Y_2^v,\overset{*}{\nabla}_{X_2^v}Z_1^h)=f_1^hg_2(X_2,Y_2)^vZ_1(f_1)^h, \end{cases}$$

We add these equations and obtain

$$A(X_i^I,Y_i^I,Z_j^J)=0, \qquad \forall (i,I),(j,J)\in\{(i,h),(2,v)\}.$$

Hence the same applies for $A(X_j^J,Y_i^I,Z_i^I)=A(X_i^I,Y_j^J,Z_i^I)=0$.
This proves that $\overset{*}{\nabla}$ is conjugate to ∇ with respect to $\tilde g_{f_1f_2}$.

□

We recall that the connection ∇ on $M_1\times M_2$ induced by $\overset{1}{\nabla}$ and $\overset{2}{\nabla}$ on M_1 and M_2 respectively, is given by Equation (24).

Proposition 7. *$(M_1,\overset{1}{\nabla},g_1)$ and $(M_2,\overset{2}{\nabla},g_2)$ are statistical manifolds if and only if $(M_1\times M_2,\tilde g_{f_1f_2},\nabla)$ is a statistical manifold.*

Proof. Let us assume that $(M_i,\overset{i}{\nabla},g_i)$ $(i=1,2)$ is statistical manifold.
Firstly, we show that ∇ is torsion-free. Indeed; by Equation (24), we have for any $X,Y\in\Gamma(TM_1\times M_2)$

$$d\pi_i(T(X,Y))=\overset{\pi_i}{\nabla}_Xd\pi_i(Y)-\overset{\pi_i}{\nabla}_Yd\pi_i(X)-d\pi_i([X,Y])$$

Since for $i=1,2$, $\overset{i}{\nabla}$ is torsion-free, then

$$\overset{\pi_i}{\nabla}_Xd\pi_i(Y)-\overset{\pi_i}{\nabla}_Yd\pi_i(X)=d\pi_i([X,Y])$$

Therefore, from Remark 1, the connection ∇ is torsion-free.
Secondly, we show that $\nabla G_{f_1,f_2}$ is symmetric. In fact; for $(i,I)\in\{(i,h),(2,v)\}$,

$$(\nabla\tilde g_{f_1f_2})(X_i^I,Y_i^I,Z_i^I)=X_i^I(\tilde g_{f_1f_2}(Y_i^I,Z_i^I))-\tilde g_{f_1f_2}(\nabla_{X_i^I}Y_i^I,Z_i^I)-\tilde g_{f_1f_2}(Y_i^I,\nabla_{X_i^I}Z_i^I)$$

by Equations (20), (24) and since $(\overset{i}{\nabla}g_i)$, $i=1,2$, is symmetric, we have

$$(\nabla\tilde g_{f_1f_2})(X_i^I,Y_i^I,Z_i^I)=(\nabla\tilde g_{f_1f_2})(Y_i^I,X_i^I,Z_i^I).$$

In the different lifts, for all $(i,I),(j,J)\in\{(i,h),(2,v)\}$, we have

$$(\nabla\tilde g_{f_1f_2})(X_i^I,Y_i^I,Z_{3-i}^J)=(\nabla\tilde g_{f_1f_2})(X_{3-i}^J,Y_i^I,Z_i^I)=(\nabla\tilde g_{f_1f_2})(X_i^I,Y_{3-i}^J,Z_i^I)=0.$$

Therefore, $(\nabla\tilde{g}_{f_1f_2})$ is symmetric. Thus $(M_1 \times M_2, \tilde{g}_{f_1f_2}, \nabla)$ is a statistical manifold.

Conversely, if $(M_1 \times M_2, \tilde{g}_{f_1f_2}, \nabla)$ is statistical manifold, then $(\nabla\tilde{g}_{f_1f_2})$ is symmetric and ∇ is torsion-free, particularly, when $X_i, Y_i, Z_i \in \Gamma(TM_i)$, we have

$$\begin{cases} (\nabla\tilde{g}_{f_1f_2})(X_i^I, Y_I^I, Z_i^I) &= (\nabla\tilde{g}_{f_1f_2})(Y_i^I, X_I^I, Z_i^I), \\ T(X_i^I, Y_i^I) &= 0. \end{cases} \quad \forall\ i = 1, 2,$$

Then, by Equations (20) and (24), we obtained, for $i = 1, 2$, $\overset{i}{\nabla} g_i$, is symmetric and $\overset{i}{\nabla}$, is torsion-free. Therefore, $(M_i, \overset{i}{\nabla}, g_i)$, $i = 1, 2$, is statistical manifold. □

At first, note that $(M_1 \times M_2, \tilde{g}_{f_1f_2}, \nabla)$ is the statistical manifold induced from $(M_1, g_1, \overset{1}{\nabla})$ and $(M_2, g_2, \overset{2}{\nabla})$.
Now, let $(M_1, \overset{1}{\nabla}, g_1)$ and $(M_2, \overset{2}{\nabla}, g_2)$ be two statistical manifolds and let $\overset{1}{\mathcal{R}}$, $\overset{2}{\mathcal{R}}$ and $\mathcal{R}$ be the curvature tensors with respect to $\overset{1}{\nabla}$, $\overset{2}{\nabla}$ and ∇ respectively.

Proposition 8. *Let $(M_i, \overset{i}{\nabla}, \overset{i}{\nabla}{}^*, g_i)$, $(i = 1, 2)$ be a connected statistical manifold. Assume that the gradient of f_i is parallel with respect to $\overset{i}{\nabla}$ and $\overset{1}{\nabla}{}^*(i = 1, 2)$. Then for any $X_i, Y_i, Z_i \in \Gamma(TM_i)$ $(i = 1, 2)$ we have*

1. $\mathcal{R}({X_1}^h, {Y_1}^h){Z_1}^h = (\mathcal{R}^1(X_1, Y_1)Z_1)^h$,
2. $\mathcal{R}({X_2}^v, {Y_2}^v){Z_2}^v = (\mathcal{R}^2(X_2, Y_2)Z_2)^v - \frac{b_1}{1+(cf_2^v)^2 b_1}\left\{(X_2 \wedge_{g_2} Y_2)Z_2\right\}^v$
$$+\frac{c^2 f_1^h f_2^v b_1}{\left(1+(cf_2^v)^2 b_1\right)^2}\left\{\left((X_2 \wedge_{g_2} Y_2)Z_2\right)(f_2)\right\}^v (gradf_1)^h,$$
3. $\mathcal{R}({X_1}^h, {Y_1}^h){Z_2}^v = 0$,
4. $\mathcal{R}({X_1}^h, {Y_2}^v){Z_1}^h = \frac{c^2 X_1(\ln f_1)^h Z_1(\ln f_1)^h Y_2(f_2)^v}{1+(cf_2^v)^2 b_1}(gradf_2)^v$,

where the wedge product $(X_2 \wedge_{g_2} Y_2)Z_2 = g_2(Y_2, Z_2)X_2 - g_2(X_2, Z_2)Y_2$.

Proof. Long but straightforward calculations as in proof of the proposal (2), where it uses the fact that connections are compatible with the metric. We obtained the same results as in (2), knowing we use only the connections are symmetrical. □

Corollary 2. *Let $(M_i, \overset{i}{\nabla}, \overset{1}{\nabla}{}^*, g_i)$, $(i = 1, 2)$ be a connected statistical manifold. Assume that f_1 is a non-constant positive function and $c \neq 0$.*
If $(\nabla, \nabla^*, \tilde{g}_{f_1f_2})$ is a dually flat structure then $(\overset{1}{\nabla}, \overset{1}{\nabla}{}^*, g_1)$ is also dually flat and $(\overset{2}{\nabla}, \overset{2}{\nabla}{}^*, g_2)$ has a constant sectional curvature.

Proof. Let $(\nabla, \nabla^*, \tilde{g}_{f_1f_2})$ be a dually flat structure.
By1. of Proposition 8, for any $X_1, Y_1, Z_1 \in \Gamma(TM_1)$, we have

$$\overset{1}{\mathcal{R}}(X_1, Y_1)Z_1 = 0,$$

From Equation (7), Since $(M_1, \overset{1}{\nabla}, g_1)$ $(i = 1, 2)$ is a statistical manifold, we have

$$\mathcal{R}^{1*}(X_1, Y_1)Z_1 = 0.$$

Hence $(M_1, \overset{1}{\nabla}, \overset{1}{\nabla}^*, g_1)$ is dually flat.
By 4. of Proposition 8, for any $X_1, Z_1 \in \Gamma(TM_1)$ and $Y_2 \in \Gamma(TM_2)$, we have

$$\frac{c^2 X_1(\ln f_1)^h Z_1(\ln f_1)^h Y_2(f_2)^v}{1+(cf_2^v)^2 b_1}(gradf_2)^v = 0.$$

So f_2 is a constant function since f_1 is non-constant function and M_2 is assumed to be connected.
Moreover, By 2. of Proposition 8, for any $X_2, Y_2, Z_2 \in \Gamma(TM_2)$, we have

$$\overset{2}{\mathcal{R}}(X_2, Y_2)Z_2 = \frac{b_1}{1+(cf_2^v)^2 b_1}\left\{(X_2 \wedge_{g_2} Y_2)Z_2\right\}^v ,$$

Since b_1 and f_2 are constants, it follows from the previous equality that $(\overset{2}{\nabla}, \overset{2}{\nabla}^*, g_2\ )$ has a constant sectional curvature $\frac{b_1}{1+(cf_2^v)^2 b_1}$. □

Acknowledgement: A big part of this work was done at The Raphel Salem Laboratory of mathematics, University of Rouen (France), Rafik Nasri and Djelloul Djebbouri would like to thank Raynaud de Fitte Paul, Siman Raulot for very useful discussions and the Mathematic section for their hospitality.

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Received xxxx 20xx; revised xxxx 20xx.

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